Symmetry breaking from Scherk-Schwarz compactification

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ABSTRACT: We analyze the classical stable configurations of an extra-dimensional gauge theory, in which the extra dimensions are compactified on a torus. Depending on the particular choice of gauge group and the number of extra dimensions, the classical vacua compatible with four-dimensional Poincaré invariance and zero instanton number may have zero energy. For $SU(N)$ on a two-dimensional torus, we find all possible degenerate zero-energy stable configurations in terms of continuous or discrete and continous parameters, for the case of trivial or non-trivial 't Hooft non-abelian flux, respectively. For the latter, an interesting rank-lowering symmetry breaking mechanism follows, in contrast to the usual Scherk-Schwarz mechanism with trivial 't Hooft flux. We explicitely describe the residual symmetries of each vacua.

Contents

1. Introduction

In the Standard Model, the explanation of the electroweak symmetry breaking pattern and the correlated hierarchy problem still remains an open issue. An intriguing extension is to consider a theory with extra space-like dimensions compactified on a non-simply connected manifold. The compactification on non-simply connected manifolds, indeed, offers a new possibility of symmetry breaking: the Scherk-Schwarz (SS) mechanism [1]. The non-local nature of this symmetry breaking protects the theory from ultraviolet divergences and makes it a promising candidate mechanism to break the electroweak symmetry.

This idea has been widely investigated in the models of gauge-Higgs unifications [2] in five- [3] and six- [4] dimensional orbifold compactification, where the four dimensional scalars are identified with the internal components of a higher-dimensional gauge field.

From the field theory point of view, a different and less explored possibility is to recover the idea of gauge-Higgs unification in the context of *flux compactification*: compactification of the extra space-like dimensions on a manifold in which there exist a (gauge) background with a non-trivial field strength, compatible with SS periodicity conditions. Such type of compactification provides an alternative tool to obtain four-dimensional chirality [5]. We investigate here some possible gauge symmetry breaking patterns that can be achieved in this context.

One simple example would be to consider a six-dimensional $U(N)$ gauge theory with the two extra dimensions compactified on a two-dimensional torus T^2. As it is well known, the presence of a stable magnetic background associated with the abelian subgroup $U(1) \in U(N)$ and living only on the two extra dimensions, induces chirality. Furthermore, it affects the non-abelian subgroup $SU(N) \in U(N)$, giving rise to a non-trivial *'t Hooft non-abelian flux* [6]. The latter can induce rich symmetry breaking patterns. While the case of trivial

non-abelian 't Hooft flux is well-known in the literature [7], the field theory analysis and the phenomenology of the non-trivial non-abelian 't Hooft flux has not been explored yet, except for a recent paper [8] analyzing mainly the case of SU(2).

The main purpose of this paper is to find and classify all possible vacua and to describe the residual symmetries for an effective four-dimensional theory obtained from a SU(N) gauge theory on a six-dimensional space-time where the two extra dimensions are compactified on a torus, for both the cases of trivial and non-trivial 't Hooft non-abelian flux. More in detail, in section 2 we provide a novel method to analyze the vacuum energy of a general Lie group on an even-dimensional torus. For the case of SU(N) on T^2, we re-obtain a well-known result [9]: the stable vacua have always zero energy, including the case with coordinate-dependent periodicity conditions. In section 3, we discuss the relation between coordinate-dependent and constant transition functions and we find under which conditions they are equivalent. For SU(N), such result will allow to introduce in section 4 what we denote as background *symmetric gauge*. In this gauge, we find and classify all the stable vacua and describe their symmetries for the case of trivial 't Hooft non-abelian flux as well as for the non-trivial case. Finally, in section 5, we conclude.

2. Vacuum energy of SU(N) on T^2

Consider a SU(N) gauge theory on a $\mathcal{M}_4 \times T^2$ space-time. In what follows, we will denote by x the coordinates of the four-dimensional Minkowski space $\mathcal{M}_4$ and by y the extra space-like dimensions.

A gauge field living on T^2 has to be periodic up to a gauge transformation under the fundamental shifts $\mathcal{T}_a : y \to y + l_a$ with $a = 1, 2$, that define the torus:[1]

$$\mathbf{A_M}(x, y + l_a) = \Omega_a(y)\mathbf{A}_M(x,y)\Omega_a^\dagger(y) + \frac{i}{g}\Omega_a(y)\partial_M\Omega_a^\dagger(y) \tag{2.1}$$

$$\mathbf{F}_{MN}(x, y + l_a) = \Omega_a(y)\mathbf{F}_{MN}(x,y)\Omega^\dagger(y), \tag{2.2}$$

where $M, N = 0, 1, \ldots, 5$, $a = 1, 2$ and l_a is the length of the direction a. The eqs. (2.1)–(2.2) are known as coordinate dependent Scherk-Schwarz compactification. The transition functions $\Omega_a(y)$ are the embedding of the fundamental shifts in the gauge space and in order to preserve four-dimensional Poincaré invariance, they can only depend on the extra dimensions y. Under a gauge transformation $S \in$ SU(N), the $\Omega_a(y)$ transform as

$$\Omega_a'(y) = S(y + l_a)\, \Omega_a(y)\, S^\dagger(y). \tag{2.3}$$

The transition functions are constrained by the following consistency condition coming from the geometry:

$$\Omega_1(y + l_2)\, \Omega_2(y) \,=\, e^{2\pi i \frac{m}{N}}\, \Omega_2(y + l_1)\, \Omega_1(y). \tag{2.4}$$

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[1]For simplicity, we consider an orthogonal torus, but all the results can be generalized to a non orthogonal T^2.

The factor $\exp[2\pi i m/N]$ is the representation of the identity in the gauge space.[2] The gauge invariant quantity $m = 0, 1, \ldots, N-1$ (modulo N) is a topological quantity called *non-abelian 't Hooft flux* [6]. In addition to the gauge transformations, the non-abelian 't Hooft flux is also invariant under the following group of transformations:

$$\Omega_a(y) \rightarrow \Omega'_a(y) \equiv z_a \Omega_a(y), \tag{2.5}$$

where z_a are elements of the center of $SU(N)$. This group is isomorphic to $(\mathbb{Z}_N)^2$.

The total Hamiltonian for a $SU(N)$ theory on a $\mathcal{M}_4 \times T^2$ space-time, reads

$$\begin{aligned}
H &= \frac{1}{2} \int_{\mathcal{M}_4} d^4x \int_{T^2} d^2y \ \mathrm{Tr}\left[\mathbf{F}^{MN}\mathbf{F}_{MN}\right] \\
&= \frac{1}{2} \int_{\mathcal{M}_4} d^4x \int_{T^2} d^2y \, \mathrm{Tr}\left[\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu} + \mathbf{F}^{\mu a}\mathbf{F}_{\mu a} + \mathbf{F}^{ab}\mathbf{F}_{ab}\right],
\end{aligned} \tag{2.6}$$

where here and in what follows, $\mu, \nu = 0, 1, 2, 3$ and a, b denote the extra coordinates. Since we are interested in configurations with $\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu} = 0$ and which preserve four-dimensional Poincaré invariance (that is $\mathbf{F}_{\mu a}(x, y) = 0$ and $\mathbf{F}_{ab}(x, y) = \mathbf{F}_{ab}(y)$), to minimize the expression in eq. (2.6) reduces to minimize the quantity

$$H_{T^2} = \frac{1}{2} \int_{T^2} d^2y \ \mathrm{Tr}\left[\mathbf{F}^{ab}(y)\,\mathbf{F}_{ab}(y)\right] \geq 0. \tag{2.7}$$

The latter inequality follows from the fact that we are working on an Euclidean manifold.

We will show that the vacuum energy is always zero, i.e. $\langle \mathbf{F}_{ab} \rangle = 0$, including the case of coordinate-dependent periodicity conditions. This result reflects the non-existence of topological quantities for a $SU(N)$ gauge theory on a T^2.

Let us consider the issue for the more general case of a Lie gauge group $\mathcal{G}$ on an even dimensional torus (T^{2n} with the integer $n \geq 1$), in order to pinpoint the dependence of the result on the choice of the gauge group and of the number of extra dimensions.

Parametrize the $(4 + 2n)$-dimensional gauge field $\mathbf{A}_M$ as

$$\begin{cases} \mathbf{A}_\mu(x, y) = A_\mu(x, y) \\ \mathbf{F}_{\mu\nu}(x, y) = F_{\mu\nu}(x, y) \end{cases}, \quad \begin{cases} \mathbf{A}_a(x, y) = B_a(y) + A_a(x, y) \\ \mathbf{F}_{ab}(x, y) = G_{ab}(y) + F_{ab}(x, y) \end{cases}, \tag{2.8}$$

where the background $B_a(y)$ has the following properties:

(i) It is a solution of the $2n$ dimensional Yang-Mills equations of motion.

(ii) It has non-trivial field strength.

(iii) It is compatible with the periodicity conditions of eqs. (2.1), (2.2).

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[2]A non-trivial value of m is possible in the absence of field representations sensitive to the center of the group.

$A_\mu(x, y)$, $A_a(x, y)$ are the fluctuations fields. The background and fluctuation field strengths are defined as

$$
\begin{aligned}
G_{ab} &= \partial_a B_b - \partial_b B_a - ig\,[B_a, B_b]\,, \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]\,, \\
F_{ab} &= D_a A_b - D_b A_a - ig\,[A_a, A_b]\,.
\end{aligned}
\tag{2.9}
$$

In eq. (2.9), $D_a A_b$ denotes the background covariant derivative

$$
D_a A_b = \partial_a A_b - ig\,[B_a, A_b]\,,
\tag{2.10}
$$

satisfying

$$
[D_a,\, D_b] = -i\,g\,G_{ab}\,.
\tag{2.11}
$$

Now $a, b = 1, \ldots, 2n$. For what follows, notice that for a non-simple gauge group, a solution of the classical Yang-Mills equations of motion can be associated to generators belonging to the normal subgroup of the algebra. Such background B_a satisfies $[B_a, A_b] = 0$ and, therefore, the covariant derivatives with respect to it reduce to ordinary derivatives.

Generalizing the discussion in ref. [10], we diagonalize the background field strength with respect to the Lorentz indices. The first step is to perform an appropriate $O(2n)$ rotation able to write the $2n \times 2n$ matrix $G_{ab}(y)$ as[3]

$$
G_{ab} =
\begin{pmatrix}
 & & & & f_1(y) & 0 & \ldots & 0 \\
 & & & & 0 & f_2(y) & \ldots & 0 \\
 & 0 & & & \ldots & \ldots & \ldots & \ldots \\
 & & & & 0 & 0 & \ldots & f_n(y) \\
-f_1(y) & 0 & \ldots & 0 & & & & \\
0 & -f_2(y) & \ldots & 0 & & & & \\
\ldots & \ldots & \ldots & \ldots & & 0 & & \\
0 & 0 & \ldots & -f_n(y) & & & &
\end{pmatrix}\,,
\tag{2.12}
$$

where $f_i(y)$ for $i = 1, \ldots, n$ are matrices belonging to the adjoint representation of the gauge group $\mathcal{G}$. The second step is to introduce the complex basis $\{z_i, \overline{z}_i\}$ defined as

$$
z_i = \frac{1}{\sqrt{2}}\,(y_i + i\,y_{n+i})\,, \qquad \overline{z}_i = \frac{1}{\sqrt{2}}\,(y_i - i\,y_{n+i})\,,
\tag{2.13}
$$

for $i = 1, \ldots, n$. In this basis, the background field strength is diagonal in the Lorentz space

$$
G_{ab} = \mathrm{Diag}\,[if_1(z), -if_1(z), if_2(z), -if_2(z), \ldots, if_n(z), -if_n(z)]\,,
\tag{2.14}
$$

and the commutators between the covariant derivatives in eq. (2.11), reduce to

$$
\begin{aligned}
\left[D_{z_i}, D_{z_j}\right] &= \left[D_{\overline{z}_i}, D_{\overline{z}_j}\right] = 0\,, \\
\left[D_{z_i}, D_{\overline{z}_j}\right] &= g\,f_i(z)\,\delta_{ij}\,, \qquad \forall\, i, j = 1, \ldots, n\,.
\end{aligned}
\tag{2.15}
$$

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[3]It follows from the fact that on an Euclidean flat space as T^{2n}, the non-trivial coordinate dependence of G_{ab} is completely determined only by the gauge indices as it can be proved using the periodicity conditions of eqs. (2.1)–(2.2) and the Yang-Mills equations of motion on a flat space.

We introduce the following gauge fixing Hamiltonian compatible with the $2n$-dimensional generalization of the periodicity conditions in eqs. (2.1), (2.2)

$$H_{g.f.} = \int_{T^{2n}} d^n z \, d^n \overline{z} \, \mathrm{Tr} \left[\sum_{i=1}^n D_{z_i} A^{\overline{z}_i} + D_{\overline{z}_i} A^{z_i} \right]^2 . \tag{2.16}$$

Denote $H_{T^{2n}}$ the $2n$-dimensional generalization of the Hamiltonian in eq. (2.7). Using eq. (2.8), the expansion of $H_{T^{2n}} + H_{g.f.}$ up to second order in the perturbation fields $A_a(x,y)$ reads

$$H_{T^{2n}} + H_{g.f.} = H^{(1A)} + H^{(2A)} + \mathcal{O}\left(A^3\right)$$

where

$$H^{(1A)} = -2 \sum_{i=1}^n \int_{T^{2n}} d^n z \, d^n \overline{z} \, \mathrm{Tr} \left[A^{z_i} D^{\overline{z}_i} G_{z_i \overline{z}_i} + A^{\overline{z}_i} D^{z_i} G_{\overline{z}_i z_i} \right] , \tag{2.17}$$

$$H^{(2A)} = \sum_{i=1}^n \int_{T^{2n}} d^n z \, d^n \overline{z} \, \mathrm{Tr} \left[A^{z_i} \mathcal{M}^2_{\overline{z}_i z_i} A^{\overline{z}_i} + A^{\overline{z}_i} \mathcal{M}^2_{z_i \overline{z}_i} A^{z_i} \right] . \tag{2.18}$$

The operators $\mathcal{M}^2_{z_i \overline{z}_i}$ and $\mathcal{M}^2_{\overline{z}_i z_i}$ in eq. (2.18) are given by

$$\mathcal{M}^2_{\overline{z}_i z_i} \equiv \sum_{k=1}^n \Sigma_k + 2\Gamma_i , \tag{2.19}$$

$$\mathcal{M}^2_{z_i \overline{z}_i} \equiv \sum_{k=1}^n \Sigma_k - 2\Gamma_i , \tag{2.20}$$

where

$$\Sigma_i \equiv - \{ D_{z_i}, D_{\overline{z}_i} \} , \tag{2.21}$$

$$\Gamma_i \equiv [D_{z_i}, D_{\overline{z}_i}] , \qquad \forall i = 1, \ldots, n . \tag{2.22}$$

The background B_a is then seen to be stable if and only if it is stationary, i.e. $H^{(1A)} = 0$, and the eigenvalues of the operators defined in eqs. (2.19), (2.20) are all semi-positive.

Since B_a is a solution of the classical equations of motion, it is stationary by construction.

In order to discuss the sign of the eigenvalues of the operators in eqs. (2.19), (2.20), we recall that

- $\forall i = 1, \ldots, n$, the operators Σ_i are defined semi-positive:

$$\Sigma_i = -D_{z_i} D_{\overline{z}_i} - D_{\overline{z}_i} D_{z_i} = |D_{z_i}|^2 + |D_{\overline{z}_i}|^2 \geq 0 , \tag{2.23}$$

since $(D_{z_i})^\dagger = -D_{\overline{z}_i}$ and $(D_{\overline{z}_i})^\dagger = -D_{z_i}$.

- The background B_a satisfies the Yang-Mills equations of motion and then the operators Σ_i, Γ_i commute. Consequently, there exists a basis that diagonalizes simultaneously (with respect to the gauge indices) these operators. We denote with $|\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\rangle$ the elements of such basis satisfying

$$\Sigma_k \ |\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\rangle = \lambda_{\Sigma_k} \ |\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\rangle \,,$$
$$\Gamma_k \ |\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\rangle = \lambda_{\Gamma_k} \ |\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\rangle \,,$$

for any $k = 1, \dots, n$.

We start analyzing the eigenvalues of the operators of eqs. (2.19)–(2.20) associated to the elements $|\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\rangle$ belonging to the subspace characterized by $\lambda_{\Gamma_i} = 0 \ \forall i = 1, \dots, n$, that is to the subspace in which $[D_{z_i}, D_{\overline{z}_i}] = 0$, $\forall i = 1, \dots, n$. All the elements of this subspace have semi-positive defined eigenvalues since eqs. (2.19)–(2.20) reduce to

$$\mathcal{M}^2_{\overline{z}_i z_i} = \mathcal{M}^2_{z_i \overline{z}_i} \equiv \sum_{k=1}^{n} \Sigma_k \geq 0 \,. \tag{2.24}$$

Notice that for the case of a non simple gauge group with background such that $[B_a, A_b] = 0$, the subspace $\lambda_{\Gamma_i} = 0$ coincides with the whole space.

Consider, now, the subspace associated to eigenvalues $\lambda_{\Gamma_i} \neq 0$. It can be analyzed using the analogy with the harmonic oscillator, i.e. using the non-trivial commutation rules in eq. (2.22). The vacuum $|0\rangle$ is characterized by

$$-D_{\overline{z}_i} D_{z_i} |0\rangle = 0 \qquad \text{if} \ \ \lambda_{\Gamma_i} < 0$$
$$-D_{z_i} D_{\overline{z}_i} |0\rangle = 0 \qquad \text{if} \ \ \lambda_{\Gamma_i} > 0 \tag{2.25}$$
$$-D_{z_i} D_{\overline{z}_i} |0\rangle = -D_{\overline{z}_i} D_{z_i} |0\rangle = 0 \qquad \text{if} \ \ \lambda_{\Gamma_i} = 0 \,.$$

For simplicity, we will discuss explicitly the subspace associated to the elements for which all $\lambda_{\Gamma_i} \neq 0$ are positive.[4] The vacuum is, therefore, defined

$$-D_{z_i} D_{\overline{z}_i} |0\rangle = 0 \,, \tag{2.26}$$

for all i associated to $\lambda_{\Gamma_i} \geq 0$. Introduce the notation $\Sigma_i |0\rangle = \lambda^0_{\Sigma_i} |0\rangle$ and $\Gamma_i |0\rangle = \lambda^0_{\Gamma_i} |0\rangle$. Since $-D_{z_i} D_{\overline{z}_i} = 1/2 \, (\Sigma_i - \Gamma_i)$, eq. (2.26) implies

$$\lambda^0_{\Sigma_i} = \lambda^0_{\Gamma_i} \,. \tag{2.27}$$

The eigenvalues of the operators in eqs. (2.19)–(2.20) associated to the vacuum $|0\rangle$ read

$$\mathcal{M}^2_{\overline{z}_i z_i} |0\rangle = \left(\sum_{k=1}^{n} \lambda^0_{\Sigma_k} + 2\lambda^0_{\Gamma_i} \right) |0\rangle \,, \tag{2.28}$$

$$\mathcal{M}^2_{z_i \overline{z}_i} |0\rangle = \left(\sum_{k=1}^{n} \lambda^0_{\Sigma_k} - 2\lambda^0_{\Gamma_i} \right) |0\rangle \,. \tag{2.29}$$

---

[4]The subspaces associated to eigenstates for which some $\lambda_{\Gamma_i} < 0$ can be obtained from the following reasoning interchanging $z_i \leftrightarrow \overline{z}_i$ for those indices i such that $\lambda_{\Gamma_i} < 0$.

Since $\lambda_{\Sigma_k} \geq 0$ for any $k = 1, \ldots, n$, the right hand side (r.h.s.) of eq. (2.28) is always positive for $\lambda_{\Gamma_i} > 0$. On the contrary, the sign of the eigenvalue in the r.h.s. of eq. (2.29) is not determined *a priori* for the general case of a Lie gauge group $\mathcal{G}$ on T^{2n}.

Focusing on the case of SU(N) on T^2, that is $n = 1$, eq. (2.29) reduces to

$$\mathcal{M}^2_{z\bar{z}} |0\rangle = -\lambda^0_\Gamma |0\rangle \qquad \text{with} \quad \lambda^0_\Gamma > 0. \qquad (2.30)$$

In this case, a background with a non-trivial field strength is, therefore, always unstable, since the operators defined in eqs. (2.19)–(2.20) always admit at least one negative eigenvalue. On the other side, all stable background configurations necessarily must have zero field strength, i.e. $\lambda_\Gamma = 0$.

Notice that such result depends on the choice of the gauge group (SU(N)) and of the number of dimensions of the torus (T^2).

Change for example the gauge group, considering instead U(N) on a T^2. U(N) is a non-simple group and, as we have discussed before, it is possible to consider solutions of the equations of motion with non-trivial field strength pointing to the internal direction associated to the identity. In this case the background covariant derivatives defined in eq. (2.10), reduce to the ordinary ones and consequently they commute. The operators $\mathcal{M}^2_{z\bar{z}}$, $\mathcal{M}^2_{\bar{z}z}$ are then given by the expressions in eq. (2.24) and, therefore, are semi-positive defined. In this case, it is, therefore, possible to have stable background with non-trivial field strength. Notice that these stable configurations have non-zero energy and are classified by some non-trivial topological charge: in this case the first Chern class.

Change now, instead, the number of dimensions of the torus. Consider for example SU(N) on T^4 ($n = 2$). In this case, eq. (2.29) reduces to

$$\mathcal{M}^2_{z_1\bar{z}_1} |0\rangle = \left(\lambda^0_{\Gamma_2} - \lambda^0_{\Gamma_1}\right) |0\rangle$$
$$\mathcal{M}^2_{z_2\bar{z}_2} |0\rangle = \left(\lambda^0_{\Gamma_1} - \lambda^0_{\Gamma_2}\right) |0\rangle.$$

Unlike for SU(N) on T^2, it is possible to have non-negative eigenvalues if the relation

$$\lambda^0_{\Gamma_1} = \lambda^0_{\Gamma_2} \qquad (2.31)$$

is fulfilled. Changing the number of torus dimensions, stable background configurations with non-trivial field strength can thus exist [11, 12]. Notice though, that although the background field strength is non-trivial, the energy can be zero. The stable configurations with non-zero energy are classified by some non zero topological charge: in this particular case, the second Chern class.

3. Coordinate dependent vs constant transition functions

In the previous section, we have provided a novel demonstration of the fact that, on a two-dimensional torus, only non-simple gauge groups admit stable configurations with non-zero energy. In particular, for the case of SU(N) on T^2 we have shown that all stable

configurations are *flat connections*, that is configurations characterized by $\mathbf{F}_{ab} = 0$ and thus zero energy. A flat connection is a pure gauge configuration[5] given by

$$B_a = \frac{i}{g} U(y) \partial_a U^\dagger(y) \,. \tag{3.1}$$

The problem of finding the non-trivial vacuum of the theory reduces then to build a $SU(N)$ gauge transformation $U(y)$ compatible with the periodicity conditions. Substituting eq. (3.1) into eq. (2.1), it follows that $U(y)$ has to satisfy

$$U(y + l_a) = \Omega_a(y) \, U(y) \, V_a^\dagger \,, \tag{3.2}$$

where $\Omega_a(y)$ are the transition functions solution of eq. (2.4), while the V_a's are constant elements of $SU(N)$ constrained by the consistency conditions

$$V_1 V_2 = e^{2\pi i \frac{m}{N}} V_2 V_1 \,. \tag{3.3}$$

Notice that given the transition functions $\Omega_1(y), \Omega_2(y)$, for each *non-physically-equivalent* pair of V_1, V_2 there exists a different gauge transformation $U(y)$ satisfying eq. (3.2) and, therefore, a different zero-energy background B_a. Here, *physically-equivalent* means that V_1, V_2 and V_1', V_2' are connected by a $SU(N)$ gauge transformation which leaves invariant $\Omega_1(y), \Omega_2(y)$.

In this section, we investigate the conditions (choice of the gauge group, number of space-like dimensions) which guarantee that eq. (3.2) admit <u>always</u> a solution regardless of the choice of Ω_a and V_a. We leave for the next section the task of classifying and describing all *non-physically-equivalent* pairs of V_a.

As in the previous section, the proof will be carried through for the general case of a Lie gauge group $\mathcal{G}$ and a T^{2n} manifold. In this case, the 't Hooft consistency conditions read

$$\Omega_a(y + l_b) \, \Omega_b(y) \,=\, \mathcal{Z}_{ab} \, \Omega_b(y + l_a) \, \Omega_a(y) \,, \tag{3.4}$$

where $\mathcal{Z}_{ab}$ is the embedding of the identity in the gauge space, that is:

$$\forall g \in \mathcal{G} \qquad \mathcal{Z}_{ab} \, g \,=\, g \, \mathcal{Z}_{ab} \,=\, g \,. \tag{3.5}$$

Since Ω_a have to commute up to a factor that plays the role of identity, it follows from eq. (3.4) that the transition functions Ω_a have to satisfy the following periodicity conditions:

$$\Omega_a(y + l_b) \,=\, g_a^{(b)}(y) \, \Omega_a(y) \,, \tag{3.6}$$

where the phases $g_a^{(b)}(y)$ are constrained to verify

$$g_b^{(a)\,-1}(y) \, g_a^{(b)}(y) \,=\, \mathcal{Z}_{ab} \,. \tag{3.7}$$

For a gauge group $\mathcal{G}$ on a $2n$-dimensional torus, all sets of transition functions, solutions of 't Hooft consistency conditions, are gauge-equivalent if and only if the group $\mathcal{G}$ is $(2n-1)$-connected, i.e. the first $(2n-1)$ homotopy groups of $\mathcal{G}$ are trivial: $\Pi_i(\mathcal{G}) = 0$.

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[5]Here we adopt the same approach and notation used in ref. [8].
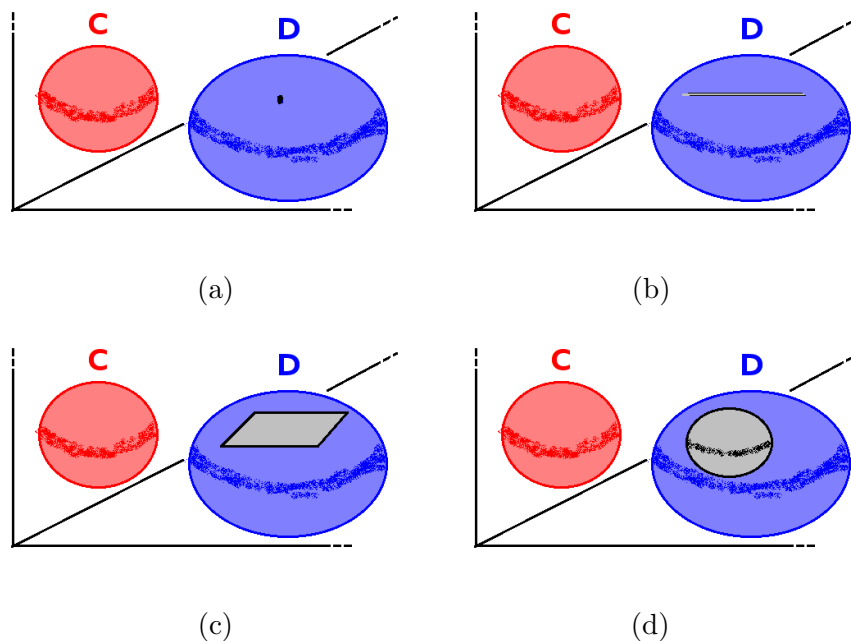
Figure 1: Examples of 3-dimensional topological spaces containing 0, 1, 2 and 3−dimensional defects (holes). In all cases, the presence of holes avoids to obtain the 3-dimensional loop **C** from the 3-dimensional loop **D** by continuous deformations. Notice that, although in the case (a) **D** can be continuously deformed to a point, the latter does not belong to the space.

Proof: Let $\Omega_a, \Omega_a^0 \in \mathcal{G}$, $a = 1, \ldots, 2n$, be generic (constant or not) sets of solutions of the consistency conditions in eq. (3.4). Considering $\mathcal{G}$ as a topological space, the transition functions Ω_a (Ω_a^0) can be seen as $2n$ points describing a $(2n-1)$-dimensional loop $\mathcal{L}_{2n}$ ($\mathcal{L}_{2n}^0$) in that space as a consequence of the constraint coming from eq. (3.4).

To understand if two sets of solutions of the 't Hooft consistency conditions are gauge-equivalent, it is tantamount to determine when $\mathcal{L}_{2n}$ can be obtained from $\mathcal{L}_{2n}^0$ by a continuous deformation, i.e. when $\mathcal{L}_{2n}$ and $\mathcal{L}_{2n}^0$ are homotopic. In particular, all $2n$-dimensional loops, contained in a topological space, belong to the same homotopy class *if and only if* they can *always* be shrunk to a point, see figure 1. This result implies that the gauge group $\mathcal{G}$ as a topological space has to be $(2n-1)$-connected: $\Pi_i(\mathcal{G}) = 0$, $\forall i = 1, \ldots, 2n-1$.

The previous reasoning can be re-formulated in a more precise way as follows. We consider the product of transformations changing step by step a given Ω_a into a Ω_a^0:

$$U(y) = \prod_{r=1}^{2n} U_r(y). \tag{3.8}$$

By construction, therefore, $U_1(y) \in \mathcal{G}$ transforms $\Omega_1 \to \Omega_1^0$, $U_2(y) \in \mathcal{G}$ transforms $U_1(y + l_2)\,\Omega_2 U_1^\dagger(y) \to \Omega_2^0$ and leaves invariant Ω_1^0, $U_3(y) \in \mathcal{G}$ transforms $U_2(y + l_3)\,U_1(y + l_3)\,\Omega_3 U_1^\dagger(y)\,U_2^\dagger(y) \to \Omega_3^0$ and leaves invariant Ω_1^0, Ω_2^0, etc. . .

Suppose that all U_r, with $r < \bar{r}$ and fixed $\bar{r} \in [1, 2n]$, exist regardless of the choice of Ω_a and Ω_a^0. We want to show that the existence of $U_{\bar{r}}(y)$ is necessary and sufficient condition to shrink to a point all $(\bar{r} - 1)$-loops of $\mathcal{G}$.

The transformation $U_{\overline{r}}$ is defined as the transformation that allows

$$U_{\overline{r}}(y)$$
$$\Omega_{\overline{r}} \Longrightarrow \Omega_{\overline{r}}^0 \,, \tag{3.9}$$

and that leaves invariant all Ω_r^0 with $r < \overline{r}$. Such a gauge transformation has to satisfy the following periodicity conditions

$$U_{\overline{r}}(y + l_r) = \Omega_r^0(y)\, U_{\overline{r}}(y)\, \Omega_r^{0\,-1}(y) \,, \qquad\qquad \forall \ r < \overline{r} \tag{3.10}$$
$$U_{\overline{r}}(y + l_{\overline{r}}) = \Omega_{\overline{r}}^0(y)\, U_{\overline{r}}(y)\, \Omega_{\overline{r}}^{-1}(y)\,. \tag{3.11}$$

To simplify the notation in what follows, let us define

$$s \equiv \{s_1, s_2, \ldots, s_{\overline{r}-1}\} \equiv \{y_1, \ldots, y_{\overline{r}-1}\} \,,$$
$$t \equiv y_{\overline{r}} \,, \tag{3.12}$$
$$u \equiv \{y_{\overline{r}+1}, \ldots, y_{2n}\} \,,$$

in such a way that $y = \{y_1, y_2, y_3, \ldots, y_{2n}\} \equiv \{s, t, u\}$. In addition, we denote with $I^{\overline{r}-1}$, the $(\overline{r}-1)$-cube defined as

$$I^{\overline{r}-1} \equiv \{(s_1, \ldots, s_{\overline{r}-1}) \,|\, 0 \leq s_i \leq l_i \ (1 \leq i \leq \overline{r}-1)\} \,, \tag{3.13}$$

and by $\partial I^{\overline{r}-1}$ the boundary of $I^{\overline{r}-1}$, defined as

$$\partial I^{\overline{r}-1} \equiv \left\{(s_1, \ldots, s_{\overline{r}-1}) \in I^{\overline{r}-1} \,|\, \text{some } s_i = 0 \text{ or } l_i\right\} \,. \tag{3.14}$$

A possible choice compatible with the periodicity condition in eq. (3.11) is

$$U_{\overline{r}}(s, 0, u) = 1 \equiv \mathcal{C}(s, u)$$
$$U_{\overline{r}}(s, l_{\overline{r}}, u) = \Omega_{\overline{r}}^0(s, 0, u)\, \Omega_{\overline{r}}^{-1}(s, 0, u) \equiv \mathcal{D}(s, u)\,. \tag{3.15}$$

Using the consistency conditions in eq. (3.4), the periodicity conditions in eq. (3.6) and the constraints in eq. (3.7), it is possible to prove that the choice in eq. (3.15) satisfies the periodicity conditions in eq. (3.10). Furthermore, it is easy to check that for $r < \overline{r}$, it results

$$\mathcal{D}(s + l_r, u) = \Omega_r^0\, \mathcal{D}(s, u)\, \Omega_r^{0\,-1} = \mathcal{D}(s, u)\,. \tag{3.16}$$

$\mathcal{C}(s, u)$ and $\mathcal{D}(s, u)$ are two $(\overline{r}-1)$-loops $\mathcal{C}, \mathcal{D} : I^{\overline{r}-1} \times T^{2n-\overline{r}} \to \mathcal{G}$ with base point $g_{\mathcal{C}}, g_{\mathcal{D}} \in \mathcal{G}$ respectively. They map, indeed, all points of the boundary $\partial I^{\overline{r}-1}$ into $g_{\mathcal{C}}, g_{\mathcal{D}} \in \mathcal{G}$ respectively:

$$\mathcal{C}(s|_{\partial I^{\overline{r}-1}}, u) = g_{\mathcal{C}} \equiv 1 \,,$$
$$\mathcal{D}(s|_{\partial I^{\overline{r}-1}}, u) = g_{\mathcal{D}} \equiv \Omega_{\overline{r}}^0(0, 0, u)\, \Omega_{\overline{r}}^{-1}(0, 0, u)\,. \tag{3.17}$$

To determine the existence of a gauge transformation $U_{\overline{r}}(y) \in \mathcal{G}$ satisfying eq. (3.15), is therefore tantamount to verify that the $(\overline{r}-1)-$loops $\mathcal{C}(s, u)$ and $\mathcal{D}(s, u)$ are homotopic.

Two $(\overline{r} - 1)-$loops $\mathcal{C}(s, u)$ and $\mathcal{D}(s, u)$ are homotopic (regardless of the choice of Ω_a^0 and Ω_a) if and only if the $(\overline{r} - 1)$-th homotopic group of $\mathcal{G}$ is trivial. The existence of the transformation $U_{\overline{r}}(y)$ guarantees, therefore, that all $(\overline{r} - 1)$-loops of $\mathcal{G}$ can be shrunk to a point.

Finally, the existence of $U(y)$ defined in eq. (3.8) $\forall \Omega_a^0$ and Ω_a is, therefore, necessary and sufficient for $\mathcal{G}$ to be $(2n - 1)$-connected.

Summarizing, we have shown that depending on the gauge group $\mathcal{G}$ and on the number of dimensions of the torus, eq. (3.2) may admit solution independently on the choice of Ω_a and V_a, satisfying eq. (2.4) and eq. (3.3), respectively. For example, for a SU(N) gauge theory on a two-dimensional torus, since such group is simply connected (that is $\Pi_1(\mathrm{SU}(N)) = 0$), two sets of solutions of the 't Hooft consistency condition are always gauge equivalent: eq. (3.2) always admits a solution.

If we increase the number of dimensions of the torus or change the gauge group, this result does not remain necessarily valid. For example:

- SU(N) is not $3-$connected, since $\Pi_3(\mathrm{SU}(N)) = \mathbf{Z}$. In consequence, if we consider SU(N) on T^4 not all the sets of transition functions can be related by a gauge transformation.

- U(N) is not simply connected and then, for U(N) on T^2, there exist solutions of the consistency conditions in eq. (3.4), inequivalent to the constant ones.

4. Vacuum symmetries and 4D spectrum

In this section, we want to find and to catalogue the possible different classical vacua for a SU(N) theory on a T^2, to discuss their symmetries and to compute the effective four-dimensional spectrum of fluctuations $\{A_\mu, A_a\}$.

In a general background gauge, such exercise can turn out to be very complicate since we have at the same time non-trivial transition functions Ω_a and non-trivial vacuum gauge configuration B_a. To simplify the discussion, it is useful to introduce the background *symmetric gauge*: the gauge in which $B_a^{\mathrm{sym}} = 0$ and $\Omega_a^{\mathrm{sym}} = V_a$.

To determine whether it is possible to go in the background *symmetric gauge* translates in to solve eq. (3.2) and therefore the SU(N) gauge transformation $S(y)$ that allows to go in that gauge is simply $S(y) = U^\dagger(y)$.

In the background *symmetric gauge*, to classify the classical vacua means to find all possible constant transitions functions $\Omega_a^{\mathrm{sym}} = V_a$, solutions of 't Hooft consistency conditions. To determine the residual symmetries reduces to establish the symmetries of $\Omega_a^{\mathrm{sym}} = V_a$. In this gauge, in fact, the periodicity conditions for the fluctuations fields $A_M = \{A_\mu, A_a\}$ are given by

$$A_M(y + l_a) = V_a A_M(y) V_a^\dagger, \tag{4.1}$$

and, therefore, the residual symmetries are associated to the SU(N) generators that commute with V_a.

We divide our analysis in two cases:

- Trivial 't Hooft non-abelian flux: $m = 0$.

- Non-trivial 't Hooft non-abelian flux: $m \neq 0$.

4.1 Trivial 't Hooft flux: $m = 0$

For $m = 0$, the transition functions commute and all the classical vacuum configurations are degenerate in energy with the trivial $\mathrm{SU}(N)$ symmetric vacuum. V_a can be parametrized as

$$V_a = e^{2\pi i \alpha_a^j H_j} , \tag{4.2}$$

where H_j are the $N - 1$ generators of the Cartan subalgebra of $\mathrm{SU}(N)$. V_a, and therefore the vacua, are characterized by $2(N - 1)$ real continuous parameters α_a^j, $0 \leq \alpha_a^j < 1$. α_a^i are non-integrable phases, which arise only in a topologically non-trivial space and cannot be gauged-away. Their values must be dynamically determined at the quantum level: only at this level the degeneracy among the infinity of classical vacua is removed [7].

The solution with $\alpha_a^j = 0$ is the fully symmetric one. For $\alpha_a^i \neq 0$, the residual gauge symmetries are those associated with the generators that commute with V_a. As V_1 and V_2 commute, the symmetry breaking is rank-preserving and the maximal symmetry breaking pattern that can be achieved is $\mathrm{SU}(N) \longrightarrow \mathrm{U}(1)^{N-1}$.

The spectrum of the fluctuations reflects the symmetry breaking pattern and it is a function of the non-integrable phases. To give an explicit expression of that spectrum, it is useful to use the Cartan-Weyl basis for the $\mathrm{SU}(N)$ generators. In addition to the generators of the Cartan subalgebra H_j with $j = 1, \ldots, N - 1$ that satisfy

$$[H_{j_1}, H_{j_2}] = 0 \quad \forall j_1, j_2 = 1, \ldots, N - 1 , \tag{4.3}$$

we denote as E_r, $r = 1, \ldots, N^2 - N$, all other $\mathrm{SU}(N)$ generators such that

$$[H_j, E_r] = q_r^j E_r \quad \forall j = 1, \ldots, N - 1 \ \text{ and } \ \forall r = 1, \ldots, N^2 - N . \tag{4.4}$$

In this basis, the four-dimensional mass spectrum for a gauge field A_M^j belonging to the Cartan subalgebra, is the ordinary Kaluza-Klein (KK) spectrum

$$m_{(j)}^2 = 4\pi^2 \left[\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} \right] , \qquad n_1, n_2 \in \mathbf{Z} . \tag{4.5}$$

For a gauge field A_M^r associated to the generator E_r, the mass spectrum reads

$$m_{(r)}^2 = 4\pi^2 \left[\left(n_1 + \sum_{j=1}^{N-1} q_r^j \alpha_1^j \right)^2 \frac{1}{l_1^2} + \left(n_2 + \sum_{j=1}^{N-1} q_r^j \alpha_2^j \right)^2 \frac{1}{l_2^2} \right] . \tag{4.6}$$

For all $\alpha_a^j \neq 0$, the only four-dimensional gauge fields that continue to be massless are the $N - 1$ fields belonging to the Cartan subalgebra. The spectrum shows the expected maximal symmetry breaking pattern: $\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}$. Finally, notice that the classical spectra described by eqs. (4.5)–(4.6) depend on the gauge indices but do not depend on the Lorentz

ones: from the four-dimensional point of view, the scalars and the gauge bosons coming from internal and ordinary components of a higher-dimensional gauge field, respectively, are expected to be degenerate at least at the classical level. However, such degeneracy is always removed at the quantum level [13].

4.2 Non-trivial 't Hooft flux: $m \neq 0$

For $m \neq 0$, the transition functions do not commute and all stable vacuum configurations induce some symmetry breaking. Eq. (3.3) reduces to the so-called two-dimensional twist algebra [6]. The first solutions were found in refs. [14, 9]. The problem for the 4-dimensional case was addressed and solved in refs. [11, 12, 15, 16]. The most general solution up to 4 dimensions was obtained in refs. [17, 18] and the d-dimensional case was studied in refs. [19 − 21].

In order to analyze the possible constant transition functions which are solutions of the two-dimensional twist algebra, let us define the quantity[6]

$$\mathcal{K} \equiv g.c.d.\,(m, N)\,, \tag{4.7}$$

and divide the analysis in two cases: $\mathcal{K} = 1$ and $\mathcal{K} > 1$.

$\mathcal{K} = 1$

In this case, the possible solutions of eq. (3.3) are of the type [11]

$$\begin{cases} V_1 = P_{(N)}^{\alpha_1} Q_{(N)}^{\beta_1} \\ V_2 = P_{(N)}^{\alpha_2} Q_{(N)}^{\beta_2} \end{cases}, \tag{4.8}$$

where the constant $N \times N$ matrices $P_{(N)}$ and $Q_{(N)}$ are defined as

$$\begin{cases} \left(P_{(N)}\right)_{kj} = e^{-2\pi i \frac{(k-1)}{N}} e^{i\pi \frac{N-1}{N}} \delta_{kj} \\ \left(Q_{(N)}\right)_{kj} = e^{i\pi \frac{N-1}{N}} \delta_{k,j-1} \end{cases}, \tag{4.9}$$

and satisfy the conditions

$$\left(P_{(N)}\right)^N = \left(Q_{(N)}\right)^N = e^{\pi i (N-1)}$$
$$P_{(N)} Q_{(N)} = e^{\frac{2\pi i}{N}} Q_{(N)} P_{(N)}\,. \tag{4.10}$$

For $m \neq 0$ and $\mathcal{K} = 1$ we have a finite number of matrices of the type in eq. (4.8) characterized by discrete parameters α_1, α_2, β_1, $\beta_2 \in [0, N-1]$ (modulo N), which have to satisfy the consistency condition

$$\alpha_1 \beta_2 \, - \, \alpha_2 \beta_1 \, = \, \mathrm{Det} \begin{pmatrix} \alpha_1 \ \alpha_2 \\ \beta_1 \ \beta_2 \end{pmatrix} \, = \, \mathrm{Det} M \, = \, m\,. \tag{4.11}$$

---

[6]g.c.d.= great common divisor.

Notice that α_1, α_2, β_1 and β_2 cannot be simultaneously zero. The condition in eq. (4.11) is invariant under the following biunimodular transformation

$$M \to M' \equiv UMV, \tag{4.12}$$

where $U, V \in \mathrm{SL}(2, \mathbb{Z})$. The invariance of the condition in eq. (4.11) under the biunimodular transformations of eq. (4.12) implies that all different matrices in eq. (4.8) are different parametrizations of the same vacuum. The only physical parameter is the value of $\mathcal{K}$.

The constant matrices in eq. (4.8) form an irreducible representation of the two-dimensional twist algebra: they have N different eigenvalues. An irreducible representation is unique modulo gauge transformations in eq. (2.3), and multiplication of the matrices by a constant in eq. (2.5).

Now, we want to show that the periodicity conditions in eq. (4.1) with the choice in eq. (4.8) completely break the original $\mathrm{SU}(N)$ symmetry group. To prove this statement, we introduce the following basis for the generators of $\mathrm{SU}(N)$:

$$\tau^{(N)}(\Delta, k_\Delta) = \sum_{n=1}^{N} e^{2\pi i \frac{n}{N} k_\Delta} \lambda^{(N)}_{(n,n+\Delta)}, \tag{4.13}$$

where $\Delta = 0, 1, \ldots, N-1$, and $k_{\Delta=0} \equiv k_0 = 1, \ldots, N-1$ and $k_{\Delta \neq 0} = 0, 1, \ldots, N-1$. The matrices $\lambda^{(N)}_{(n,m)}$ are the $N \times N$ matrices defined by

$$\left(\lambda^{(N)}_{(n,m)}\right)_{ij} \equiv \delta_{ni}\delta_{mj}. \tag{4.14}$$

The matrices $\tau^{(N)}(\Delta, k_\Delta)$ are eigenstates of the operators $P_{(N)}$, $Q_{(N)}$ with eigenvalues $e^{2\pi i \frac{\Delta}{N}}$ and $e^{2\pi i \frac{k_\Delta}{N}}$, respectively and satisfy the following properties

$$\mathrm{Tr}\, \tau^{(N)}(\Delta, k_\Delta) = 0$$
$$\frac{1}{N}\, \mathrm{Tr}\left[\tau^{(N)\dagger}(\Delta, k_\Delta)\tau^{(N)}(\Delta', k'_\Delta)\right] = \delta_{\Delta,\Delta'}\, \delta_{k_\Delta, k'_\Delta}$$
$$\left[\tau^{(N)}(\Delta, k_\Delta),\, \tau^{(N)}(\Delta', k'_\Delta)\right] = \left(e^{\frac{2\pi i}{N}\Delta k'_\Delta} - e^{\frac{2\pi i}{N}\Delta' k_\Delta}\right) \tau^{(N)}\left(\Delta + \Delta',\, k_\Delta + k'_\Delta\right).$$

In this basis, the periodicity conditions reads

$$V_a\, \tau^{(N)}(\Delta, k_\Delta)\, V_a^\dagger = e^{2\pi i \frac{\alpha_a \Delta + \beta_a k_\Delta}{N}}\, \tau^{(N)}(\Delta, k_\Delta) \qquad \text{for} \quad a = 1, 2. \tag{4.15}$$

The residual symmetries are associated to $\mathrm{SU}(N)$ generators that commute simultaneously with V_1 and V_2, that is to those $\tau^{(N)}(\Delta, k_\Delta)$ for which it results

$$\frac{\alpha_a \Delta + \beta_a k_\Delta}{N} \in \mathbf{Z}. \tag{4.16}$$

Using the Bezout theorem, it is possible to prove that the number of $\mathrm{SU}(N)$ generators $\tau^{(N)}(\Delta, k_\Delta)$ satisfying the condition in eq. (4.16), that is the dimension of the residual symmetry group $\mathcal{H} \subset \mathrm{SU}(N)$, results

$$\mathrm{Dim}\,[\mathcal{H}] = \mathcal{K}^2 - 1. \tag{4.17}$$

For $\mathcal{K} = 1$, therefore, $\mathrm{SU}(N)$ is completely broken by the V_a in eq. (4.8).

$K > 1$.   In this case, the matrices in eq. (4.8) with the constraint in eq. (4.11) form a reducible representation of the twist algebra. Unlike the $\mathcal{K} = 1$ case, indeed, the matrices in eq. (4.8) are now invariant under a subgroup $\mathcal{H}$ of SU(N) given by

$$\mathcal{H} = \mathrm{SU}(\mathcal{K}) \subset \mathrm{SU}(N). \tag{4.18}$$

Since $\mathcal{K} < N$, the rank of $\mathcal{H}$ is always less than the one of SU(N).

For $\mathcal{K} > 1$, the irreducible representations of the twist algebra (up to gauge transformations in eq. (2.3) and multiplications by a constant as in eq. (2.5)) can be obtained in the following way

$$
\begin{aligned}
V_1 &= \omega_1 \, P^{\alpha_1} \, Q^{\beta_1} \\
V_2 &= \omega_2 \, P^{\alpha_2} \, Q^{\beta_2} \,.
\end{aligned} \tag{4.19}
$$

Now, the SU(N) constant matrices P and Q are given by

$$
P \equiv \begin{pmatrix}
P_{(N/\mathcal{K})} & 0 & \dots & 0 \\
0 & P_{(N/\mathcal{K})} & \dots & 0 \\
\dots & \dots & \dots & \dots \\
0 & 0 & \dots & P_{(N/\mathcal{K})}
\end{pmatrix}_{(\mathcal{K} \times \mathcal{K})}
$$

$$
Q \equiv \begin{pmatrix}
Q_{(N/\mathcal{K})} & 0 & \dots & 0 \\
0 & Q_{(N/\mathcal{K})} & \dots & 0 \\
\dots & \dots & \dots & \dots \\
0 & 0 & \dots & Q_{(N/\mathcal{K})}
\end{pmatrix}_{(\mathcal{K} \times \mathcal{K})} , \tag{4.20}
$$

where $P_{N/\mathcal{K}}$ and $Q_{N/\mathcal{K}}$ are $N/\mathcal{K} \times N/\mathcal{K}$ matrices defined as in eqs. (4.9)–(4.10) with the change $N \to N/\mathcal{K}$. In this case, the discrete parameters α_1, α_2, β_1, β_2 have to satisfy the constraint in eq. (4.11) where $m \to m/\mathcal{K}$. The constant matrices ω_1, ω_2 are elements of the subgroup $\mathrm{SU}(\mathcal{K}) \subset \mathrm{SU}(N)$ and have to satisfy the constraint

$$\omega_1 \, \omega_2 = \omega_2 \, \omega_1 \,. \tag{4.21}$$

As in the $m = 0$ case, ω_1 and ω_2 commute and therefore they can be parametrized in terms of generators (H_j) belonging to the Cartan subalgebra of SU($\mathcal{K}$):

$$\omega_a = e^{2\pi i \sum_{j=1}^{\mathcal{K}-1} \phi_a^j H_j} \,. \tag{4.22}$$

ϕ_a^j are $2(\mathcal{K} - 1)$ real continuous parameters taking values in the interval $0 \leq \phi_a^j < 1$ (modulo integers). They are non-integrable phases and their values must be dynamically determined at the quantum level: only at this level the degeneracy among the infinity of classical values can be removed [13].

Summarizing, in the $m \neq 0$ case the possible solutions of condition in eq. (3.3) and therefore the possible stable vacua are given by the formula in eq. (4.19) with the restriction that for $\mathcal{K} = 1$ it results $\omega_1 = \omega_2 = 1$. Such vacua are characterized by the value of $\mathcal{K}$ and by $2(\mathcal{K} - 1)$ continous ϕ_a^j parameters.

1. In the case $\phi_a^j = 0 \ \forall j$, the residual symmetry group is given by the SU(N) subgroup that commutes with the matrices P and Q in eq. (4.19). These matrices always induce some degree of symmetry breaking due to their non-trivial commutation rules. Such symmetry breaking mechanism is rank-lowering and, as discussed before, the pattern that can be achieved results SU(N) $\rightarrow$ SU($\mathcal{K}$). In particular, for $\mathcal{K} = 1$, SU(N) is completely broken.

2. For $\mathcal{K} > 1$ and non-vanishing phases ϕ_a^j, there is additional degrees of freedom with respect to the symmetry breaking pattern. Since ω_1 and ω_2 commute, such new symmetry breaking mechanism preserves the rank of SU($\mathcal{K}$), and the maximal symmetry breaking that can be achieved is

$$\mathrm{SU}(N) \rightarrow \mathrm{SU}(\mathcal{K}) \rightarrow \mathrm{U}(1)^{\mathcal{K}-1} . \tag{4.23}$$

In order to discuss the effective four dimensional mass spectrum, we need to diagonalize the constant transition functions V_a and consequently the periodicity conditions. To do that, we generalize the basis of generators of SU(N) introduced in eq. (4.13) as follows

$$\tau_{(\rho,\sigma)}(\Delta, k_\Delta) = \begin{cases} \text{if } \begin{cases} \rho = \sigma \\ \Delta = k_\Delta = 0 \end{cases} \Rightarrow \left(\sum_{i=1}^{\rho} \lambda_{(i,i)}^{(\mathcal{K})} - \rho \lambda_{(\rho+1,\rho+1)}^{(\mathcal{K})} \right) \otimes 1_{(N/\mathcal{K})} \\ \\ \text{else} \qquad\qquad \Rightarrow \lambda_{(\rho,\sigma)}^{(\mathcal{K})} \otimes \tau^{(N/\mathcal{K})}(\Delta, k_\Delta) \end{cases} \tag{4.24}$$

where now $\Delta, k_\Delta = 0, \ldots, N/\mathcal{K} - 1$ and $\rho, \sigma = 1, \ldots, \mathcal{K}$ excluding the case $\Delta = k_\Delta = 0$ and $\rho = \sigma$ in which $\rho = 1, \ldots, \mathcal{K} - 1$. Here, $\tau^{(N/\mathcal{K})}(\Delta, k_\Delta)$ are $N/\mathcal{K} \times N/\mathcal{K}$ matrices defined as in eq. (4.13), but with the change $N \rightarrow N/\mathcal{K}$. In the same way, the $\mathcal{K} \times \mathcal{K}$ matrices $\lambda_{(\rho,\sigma)}^{(\mathcal{K})}$ can be obtained from eq. (4.14) with the substitution $N \rightarrow \mathcal{K}$.

In this basis, the generators which commute with P and Q and form the subgroup SU($\mathcal{K}$), can be identified with $\tau_{\rho,\sigma}(0,0)$. In particular, the generators belonging to the Cartan subalgebra of SU($\mathcal{K}$) and appearing in eq. (4.22) are given by $H^{j \equiv \rho} = \tau_{\rho,\rho}(0,0)$.

Now, the periodicity condition read

$$V_a \tau_{(\sigma,\tau)}(\Delta, k_\Delta) V_a^\dagger = e^{2\pi i \left(\frac{\mathcal{K}}{N}(\alpha_a \Delta + \beta_a k_\Delta) + q_\rho^{(\sigma,\tau)} \phi_a^{(\rho)} \right)} \tau_{(\sigma,\tau)}(\Delta, k_\Delta) , \tag{4.25}$$

with

$$\left[\tau_{(\rho,\rho)}(0,0), \tau_{(\sigma,\tau)}(\Delta, k_\Delta) \right] = q_\rho^{(\sigma,\tau)} \tau_{(\sigma,\tau)}(\Delta, k_\Delta) . \tag{4.26}$$

At the classical level, the effective four-dimensional mass spectrum is, also in this case, independent of the Lorentz index M and takes the following form

$$m^2 = 4\pi^2 \sum_{a=1}^{2} \left(n_a + \frac{\mathcal{K}}{N}(\alpha_a \Delta + \beta_a k_\Delta) + q_\rho^{(\sigma,\tau)} \phi_a^{(\rho)} \right)^2 \frac{1}{l_a^2} , \tag{4.27}$$

with $n_1, n_2 \in \mathbf{Z}$.

The effective four-dimensional mass spectrum reflects the symmetry breaking pattern discussed before. Given α_1, α_2, β_1, β_2 and for all $\phi_a^\rho = 0$, only gauge bosons associated to the generators $\tau_{(\rho,\sigma)}(0,0)$ of SU($\mathcal{K}$) admit zero modes. Since α_1, α_2, β_1, β_2 cannot be simultaneously zero, the spectrum described by eq. (4.27) exhibits always some degree of symmetry breaking. For $\mathcal{K} > 1$ and all $\phi_a^j \neq 0$, the only massless modes arise from gauge bosons associated to the Cartan subalgebra of SU($\mathcal{K}$): $\tau_{(\rho,\rho)}(0,0)$.

Finally, it is worth to underline the different nature of the symmetry breaking for the two cases of trivial ($m = 0$) and non-trivial ($m \neq 0$) 't Hooft non-abelian flux. In the $m = 0$ case, the gauge symmetry breaking mechanism is, indeed, the Hosotani mechanism [7]: it is always possible to choose an appropriate background gauge, compatible with the consistency conditions, in which the transition functions are trivial ($V_1 = V_2 = \mathbf{1}$) and some extra space-like components of the six-dimensional gauge fields $\mathbf{A}_a$ acquire a vacuum expectation value (VEV): $\langle \mathbf{A}_a \rangle = B_a$. In this case, the symmetry breaking can be seen as spontaneous in the following sense:

1. For each 4-dimensional massive gauge field $\mathbf{A}_\mu$, there exists a linear combination of the $\mathbf{A}_a$ that play the role of a 4-dimensional scalar pseudo-goldstone boson, eaten by the 4-dimensional gauge bosons to become a longitudinal gauge degree of freedom.

2. The VEV of $\mathbf{A}_a$ works as the order parameter of the symmetry breaking mechanism. In particular, it is possible to deform $\langle \mathbf{A}_a \rangle$ to zero compatibly with the consistency conditions, so as to restore all the initial symmetries.

In the $m \neq 0$ case, we cannot interpret all steps of symmetry breaking in eq. (4.23) as spontaneous:

1. The SU(N) $\rightarrow$ SU($\mathcal{K}$) symmetry breaking is due to the constant matrices P and Q of eqs. (4.19)–(4.20). They don't commute as a consequence of the non-trivial value of the 't Hooft flux. *The symmetry breaking can not be related* only *to the VEV of* $\mathbf{A}_a$. Although for each massive 4-dimensional gauge boson $\mathbf{A}_\mu \in \frac{\text{SU}(N)}{\text{SU}(\mathcal{K})}$ there exists a 4-dimensional pseudo-goldstone boson, in this case it is not possible to determine an order parameter that can be deformed compatibly with the consistency conditions in such a way to restore all the initial symmetries.

2. The SU($\mathcal{K}$) $\rightarrow$ U(1)$^{\mathcal{K}-1}$ symmetry breaking is due to the matrices ω_1 and ω_2 that commute. This symmetry breaking mechanism exactly works as in the $m = 0$ case: it is always possible to choose an appropriate SU($\mathcal{K}$) background gauge in which $\omega_1 = \omega_2 = \mathbf{1}$ and some extra space-like components of the six-dimensional gauge fields $\mathbf{A}_a \in$ SU($\mathcal{K}$) acquire a VEV. This step can be understood as a consequence of a spontaneous symmetry breaking mechanism.

5. Conclusions

We have studied extra-dimensional gauge theories with the extra dimensions compactified *á la* Scherk-Schwarz on toroidal manifolds.

Using the analogy with the harmonic oscillator, we have analyzed the vacuum energy for a general group on an even-dimensional torus. For the particular case of SU(N) on T^2, we have re-obtained, with a novel procedure, the well-known result that all stable vacua, compatible with four-dimensional Poincaré invariance and zero four-dimensional instanton number, have zero energy.

We have, then, studied the classical zero-energy vacua, for a gauge theory on an even-dimensional torus, with periodicity conditions satisfying the 't Hooft consistency conditions. For SU(N) on T^2, the set of degenerate and inequivalent classical zero-energy vacua is completely determined by the set of *constant* transition functions V_a solutions of the 't Hooft consistency conditions. We have explicitly proved that such result depends on the particular choice of the gauge group and of the number of extra dimensions.

The number of vacua, the residual symmetries and the nature of the symmetry breaking mechanism are determined by the value of the 't Hooft non-abelian flux:

- For trivial 't Hooft flux, $m = 0$, it results a continuum of vacua, degenerate at the classical level with the SU(N) symmetric one. The symmetry breaking is rank-preserving and spontaneous since it is exactly as the Hosotani mechanism.

- The main novel result of this paper is the explicit demonstration of the symmetry breaking pattern and the four-dimensional mass spectrum, for the case of non-trivial 't Hooft flux. For $m \neq 0$, the number of vacua depends on the value of $\mathcal{K} = $ g.c.d.(m, N). For $\mathcal{K} = 1$ there exists only one classical vacuum and SU(N) is completely broken. For $\mathcal{K} > 1$, there is a degeneracy among an infinity of classical vacua and SU(N) is partially broken in all of them. In particular, the symmetry breaking pattern can be seen as due to two different mechanisms. The first induces rank-lowering symmetry breaking that, from the four-dimensional point of view, can be considered as explicit. The second mechanism, instead, gives rise to a rank-preserving symmetry breaking that can be interpreted as spontaneous.

The clarification of the rank-lowering mechanism for $m \neq 0$ case, should be of general phenomenological interest. It may overcome the phenomenological short-comings associated to the usual Scherk-Schwarz mechanism with trivial 't Hooft flux. These consequences are under study [13].

Acknowledgments

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Erratum

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